Strong duality data of type A and extended T-systems

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Fundamental notation

$\mathfrak{g}$: affine Lie algebra with index set $[0,n] := \{0,1,\ldots,n\}$

i.e. $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}K$ ($\mathfrak{g}_0$: simple Lie algebra): untwisted type

or its certain Lie subalgebra: twisted type

$U_q'(\mathfrak{g})$: **quantum affine algebra** (without a degree operator)

assoc. alg. over $\mathbf{k} = \overline{\mathbb{C}(q)}$ defined as a q-deformation of $U(\mathfrak{g})$

$\mathscr{C}_\mathfrak{g}$: the cat. of f.d. $U_q'(\mathfrak{g})$-mod. (of type 1)

- $\mathscr{C}_\mathfrak{g}$ is a monoidal category with $\otimes$ and the trivial module $\mathbf{1}$
  $\Rightarrow K(\mathscr{C}_\mathfrak{g})$ has a ring structure (**Grothendieck ring**)
- Each $M \in \mathscr{C}_\mathfrak{g}$ has the right dual $\mathscr{D}(M)$ and the left dual $\mathscr{D}^{-1}(M)$

Theorem (Chari-Pressley, 95)

$$\{\text{simples in } \mathscr{C}_{\mathfrak{g}}\}/\cong \overset{1:1}{\leftrightarrow} \{\boldsymbol{\pi}(u) = (\pi_1(u), \ldots, \pi_n(u)) \mid \pi_i(u) \in 1 + u\mathbf{k}[u]\}$$

Drinfeld polynomials

monomial parametrization

In the sequel, we always assume that each $\pi_i(u)$ is of the form

$$\pi_i(u) = (1 - q^{k_1}u) \cdots (1 - q^{k_p}u) \quad (k_r \in \mathbb{Z}).$$

$$L\Big(\prod_{i,r} Y_{i,k_r^{(i)}}\Big) \leftrightarrow \boldsymbol{\pi}(u) = \Big(\pi_i(u) = (1 - q^{k_1^{(i)}}u) \cdots (1 - q^{k_s^{(i)}}u)\Big)_{1 \le i \le n}$$

$\nwarrow$ **highest monomial**

Mukhin–Young's extended T-systems

Mukhin–Young introduced in '12 the following relations in $K(\mathscr{C}_{\mathfrak{g}})$:

If $\mathfrak{g}$: type $A_n^{(1)}$ or $B_n^{(1)}$ and $L\big(\prod_{r=1}^{p} Y_{i_r,k_r}\big)$: **prime snake module**, then

$$\Big[L\Big(\prod_{r=1}^{p-1} Y_{i_r,k_r}\Big) \otimes L\Big(\prod_{r=2}^{p} Y_{i_r,k_r}\Big)\Big] = \Big[L\Big(\prod_{r=1}^{p} Y_{i_r,p_r}\Big) \otimes L\Big(\prod_{r=2}^{p-1} Y_{i_r,k_r}\Big)\Big] + [M \otimes N]$$

This is a generalization of the T-systems for Kirillov-Reshetikhin (KR) modules, which exist for general $\mathfrak{g}$:

Ex. (T-systems for untwisted, simply-laced $\mathfrak{g}$)

$$\Big[L\Big(\prod_{k=1}^{p-1} Y_{i,2k}\Big) \otimes L\Big(\prod_{k=2}^{p} Y_{i,2k}\Big)\Big] = \Big[L\Big(\prod_{k=1}^{p} Y_{i,2k}\Big) \otimes L\Big(\prod_{k=2}^{p-1} Y_{i,2k}\Big)\Big]$$

$$+ \Big[\bigotimes_{c_{ij}=-1} L\Big(\prod_{k=1}^{p-1} Y_{j,2k+1}\Big)\Big]$$

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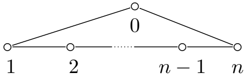
Snake modules in type $A_n^{(1)}$

Assume $\mathfrak{g}$ is of type $A_n^{(1)}$:



Set $J_A := \{(i,k) \mid k \equiv i \pmod 2\} \subseteq [1,n] \times \mathbb{Z}$

$(n = 5)$

$(i \setminus k)$	$\cdots$	0	1	2	3	4	5	6	7	$\cdots$
1			○		○		○		○	
2		○		○		○		○		
3			○		○		○		○	
4		○		○		○		○		
5			○		○		○		○	

Definition

A sequence $\xi = \big((i_1,k_1),\ldots,(i_p,k_p)\big) \in J_A^p$ is a **snake** (prime snake)

$\overset{\text{def}}{\Leftrightarrow}$ for $1 \leq {}^{\forall}r < p$, setting $(i,k) = (i_r,k_r)$ • and $(i',k') = (i_{r+1},k_{r+1})$,

$$|i-i'| + 2 \leq k - k' \ (\leq \min\{i+i', 2n+2-i-i'\})$$

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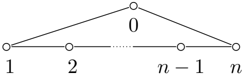

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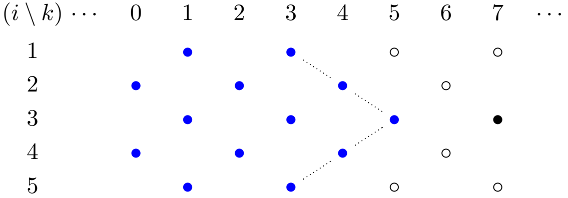

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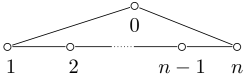

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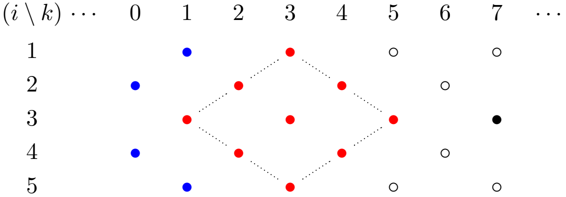

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2	●		●		○		○		○		○		○		
3		○		○		●		○		○		○		○	
4	○		○		○		○		●		○		○		
5		○		○		○		○		○		●		●	

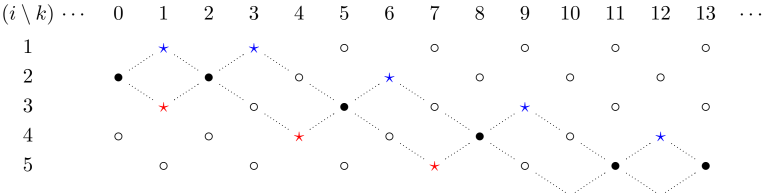
$\boldsymbol{\xi} = \big((i_1, k_1), \ldots, (i_p, k_p)\big)$: snake $\Rightarrow$ $L(\boldsymbol{\xi}) = L(\prod_{r=1}^{p} Y_{i_r,k_r})$: snake module

prime snake $\xi \rightsquigarrow$ two neighboring snakes ξ_{H} ($\star$), ξ_{L} ($\star$)

Theorem (MY12)

- ξ: prime $\Leftrightarrow$ $L(\xi)$: prime (i.e. $L(\xi) \cong M \otimes N \Rightarrow M \cong \mathbf{1}$ or $N \cong \mathbf{1}$)
- $[L(\prod_{r=1}^{p-1} Y_{i_r,k_r}) \otimes L(\prod_{r=2}^{p} Y_{i_r,k_r})] = [L(\xi) \otimes L(\prod_{r=2}^{p-1} Y_{i_r,k_r})] + [L(\xi_{\mathrm{H}}) \otimes L(\xi_{\mathrm{L}})]$

↖ simple ↗



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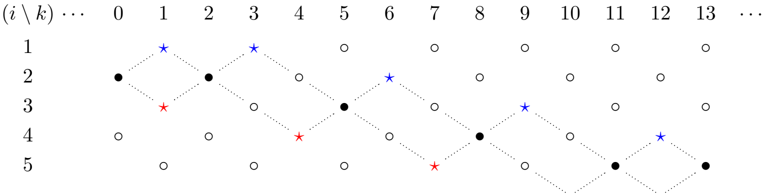

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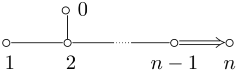
Extended T-systems in type $B_n^{(1)}$



$$J_B = \{(i,k) \mid k \equiv \delta_{in} \pmod 2\} \subseteq [1,n] \times \mathbb{Z}$$

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	1	○				○				○				○				○				○				○	
	2			○				○				○				○				○				○			
(n=3)	3		○		○		○		○		○		○		○		○		○		○		○		○		
	2	○				○				○				○				○				○				○	
	1			○				○				○				○				○				○			

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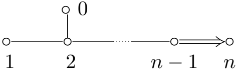

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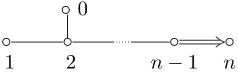

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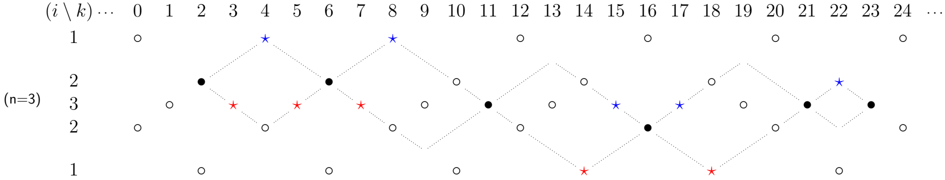

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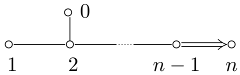

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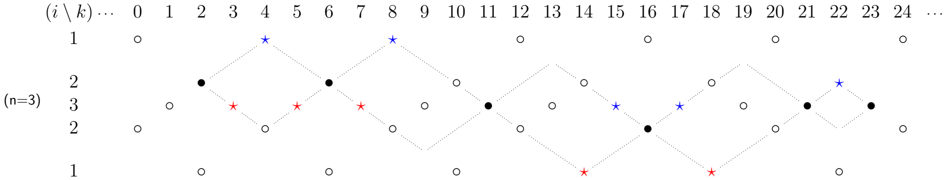

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Proof of Mukhin–Young's extended T-systems

$M \in \mathscr{C}_{\mathfrak{g}} \rightsquigarrow \mathrm{ch}_q\, M \in \mathbb{Z}_{\geq 0}[Y_{i,k}^{\pm 1} \mid i \in I, k \in \mathbb{Z}]$: **$q$-character**

(character w.r.t. a large comm. subalg. $U_q(\mathsf{L}\mathfrak{h}_0)$)

<u>Fact</u> ch_q induces an injective ring hom. $K(\mathscr{C}_{\mathfrak{g}}) \hookrightarrow \mathbb{Z}[Y_{i,k}^{\pm 1}]$.

It suffices to show that

$$\mathrm{ch}_q\, L(\boldsymbol{\xi}')\mathrm{ch}_q\, L(\boldsymbol{\xi}'') = \mathrm{ch}_q\, L(\boldsymbol{\xi})\mathrm{ch}_q\, L(\boldsymbol{\xi}''') + \mathrm{ch}_q\, L(\boldsymbol{\xi}_{\mathrm{H}})\mathrm{ch}_q\, L(\boldsymbol{\xi}_{\mathrm{L}})$$

This is proved by using the path description formula of $\mathrm{ch}_q\, L(\boldsymbol{\xi})$ [MY12b]

Questions arising from the extended T-systems

- $0 \to L(\boldsymbol{\xi}') \otimes L(\boldsymbol{\xi}'') \to L(\boldsymbol{\xi}) \otimes L(\boldsymbol{\xi}''') \to L(\boldsymbol{\xi}_{\mathrm{H}}) \otimes L(\boldsymbol{\xi}_{\mathrm{L}}) \to 0$ or
  $0 \to L(\boldsymbol{\xi}_{\mathrm{H}}) \otimes L(\boldsymbol{\xi}_{\mathrm{L}}) \to L(\boldsymbol{\xi}) \otimes L(\boldsymbol{\xi}''') \to L(\boldsymbol{\xi}') \otimes L(\boldsymbol{\xi}'') \to 0$
- Are there other modules satisfying these relations?
- Why prime snake modules satisfy these relations?
- Are there extended T-systems in other types?
  (G_2: Li–Mukhin, '13 C_3: Li, '15)

  In other types, q-characters are more complicated
  (q-char. of snake modules in $A_n^{(1)}$, $B_n^{(1)}$ are "special")
  $\rightsquigarrow$ Another approach without relying on q-characters?

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  ⇝ Another approach without relying on q-characters?

Previous research: Kashiwara–Kim–Oh–Park

$C = (c_{ij})_{i,j \in I}$: Cartan matrix of finite ADE type

$\mathcal{D} = \{\mathsf{L}_i\}_{i \in I} \subseteq \mathscr{C}_{\mathfrak{g}}$: **strong duality datum** associated with C

Fix $\boldsymbol{i} = (i_1, \dots, i_N)$: red. word of the longest element $w_0 \in W(C)$

$\rightsquigarrow S_l = S_l^{\mathcal{D},\boldsymbol{i}} \in \mathscr{C}_{\mathfrak{g}}$ ($l \in \mathbb{Z}$): **affine cuspidal modules**

Rem. For each $\mathfrak{g}$, ${}^{\exists}$a "canonical pair" $\big(\mathcal{D}(\mathfrak{g}), \boldsymbol{i}(\mathfrak{g})\big)$ (coming from a Q-datum)

s.t. $\{S_l^{\mathcal{D}(\mathfrak{g}),\boldsymbol{i}(\mathfrak{g})}\} = \{L(Y_{i,k}) \mid (i,k) \in J\}$ (fundamental modules)

For any pair $(\mathcal{D}, \boldsymbol{i})$, they defined **affine determinantial modules** $M_i[a,b]$ as the head of the tensor of certain S_k's.

(For a canonical pair $(\mathcal{D}(\mathfrak{g}), \boldsymbol{i}(\mathfrak{g}))$, $M_i[a,b]$ coincide with KR modules)

Theorem (KKOP, '22)

$M_i[a,b]$ satisfy a short exact seq. corresponding to the T-systems.

Outline of main results

$\mathcal{D} = \{\mathsf{L}_i\}_{i\in I} \subseteq \mathscr{C}_{\mathfrak{g}}$: **strong duality datum** of type A_n

Set $\boldsymbol{i}^A := \boldsymbol{i}(A_n^{(1)})$, $\boldsymbol{i}^B := \boldsymbol{i}(B_{n_0}^{(1)}) \quad (n = 2n_0 - 1)$

: red. words appearing in the canonical pairs $(\mathcal{D}(\mathfrak{g}), \boldsymbol{i}(\mathfrak{g}))$

$\rightsquigarrow S_k^A := S_k^{\mathcal{D},\boldsymbol{i}^A}$, $S_k^B := S_k^{\mathcal{D},\boldsymbol{i}^B}$ **affine cuspidal modules**

$\rightsquigarrow \mathbb{S}^A(\boldsymbol{\xi})$, $\mathbb{S}^B(\boldsymbol{\xi})$: **snake modules associated with $\mathcal{D}$**

Rem. $\mathcal{D} = \mathcal{D}(A_n^{(1)}) \Rightarrow \mathbb{S}^A(\boldsymbol{\xi})$: MY's snake mod. of $A_n^{(1)}$.
$\mathcal{D} = \mathcal{D}(B_{n_0}^{(1)}) \Rightarrow \mathbb{S}^B(\boldsymbol{\xi})$: MY's snake mod. of $B_{n_0}^{(1)}$

Theorem (N)

For $X \in \{A, B\}$,

$$0 \to \mathbb{S}^X(\boldsymbol{\xi}_{\mathrm{H}}) \otimes \mathbb{S}^X(\boldsymbol{\xi}_{\mathrm{L}}) \to \mathbb{S}^X(\boldsymbol{\xi}_{[1,p-1]}) \otimes \mathbb{S}^X(\boldsymbol{\xi}_{[2,p]}) \to \mathbb{S}^X(\boldsymbol{\xi}) \otimes \mathbb{S}^X(\boldsymbol{\xi}_{[2,p-1]}) \to 0$$

- The first and the third terms are simple.

Questions arising from the extended T-systems

- $0 \to L(\boldsymbol{\xi}') \otimes L(\boldsymbol{\xi}'') \to L(\boldsymbol{\xi}) \otimes L(\boldsymbol{\xi}''') \to L(\boldsymbol{\xi}_{\mathrm{H}}) \otimes L(\boldsymbol{\xi}_{\mathrm{L}}) \to 0$ or
  $0 \to L(\boldsymbol{\xi}_{\mathrm{H}}) \otimes L(\boldsymbol{\xi}_{\mathrm{L}}) \to L(\boldsymbol{\xi}) \otimes L(\boldsymbol{\xi}''') \to L(\boldsymbol{\xi}') \otimes L(\boldsymbol{\xi}'') \to 0$ ○
- Are there other modules satisfying these relations? ○
- Why prime snake modules satisfy these relations? △
- Are there extended T-systems in other types? ✕
  $\rightsquigarrow$ another approach without relying on q-characters?

Another quetion Relations with cluster algebras?

- All prime snake mod. corresp. to cluster variables [Duan–Li–Luo, 19]

Questions arising from the extended T-systems

- $0 \to L(\boldsymbol{\xi}') \otimes L(\boldsymbol{\xi}'') \to L(\boldsymbol{\xi}) \otimes L(\boldsymbol{\xi}''') \to L(\boldsymbol{\xi}_{\mathrm{H}}) \otimes L(\boldsymbol{\xi}_{\mathrm{L}}) \to 0$ or
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