Graded limits of minimal affinizations over a quantum loop algebra

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Abstract

$U_q(\mathfrak{g})$: quantized env. alg. of a simple Lie algebra $\mathfrak{g}$

$V_q(\lambda)$: f.d. irred. $U_q(\mathfrak{g})$-module $\overset{q\to 1}{\Rightarrow}$ $V(\lambda)$: irred. $\mathfrak{g}$-module

- $\mathrm{ch}\, V_q(\lambda) = \mathrm{ch}\, V(\lambda)$ ←well-known

$U_q(L\mathfrak{g})$: quantum loop algebra ($L\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$)

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- $\mathrm{ch}\, V_q = \mathrm{ch}\, V$
- V: possibly reducible ⇒ not easy to study the str. of V

V_q: some special modules (e.g. Kirillov-Reshetikhin module)

⇝ possible to study the str. of V [Chari, Moura].

Today V_q: Minimal affinizations of type ABC.

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Minimal affinization

$V_q(\lambda)$: irred. $U_q(\mathfrak{g})$-module (λ: dom. integral wt of $\mathfrak{g}$).

M: f.d. irred. $U_q(L\mathfrak{g})$-module is an affinization of $V_q(\lambda)$

$$\overset{\text{def}}{\iff} M \cong V_q(\lambda) \oplus \bigoplus_{\mu<\lambda} V_q(\mu)^{\oplus m(\mu)} \text{ as a } U_q(\mathfrak{g})\text{-module.}$$

M: minimal affinization of $V_q(\lambda)$

$\overset{\text{def}}{\iff}$ ∘ M: affinization of $V_q(\lambda)$,

∘ The part $\bigoplus_{\mu<\lambda} V_q(\mu)^{\oplus m(\mu)}$ is "minimal".

$$\left(\begin{array}{l} \bigoplus V_q(\mu)^{\oplus m(\mu)} \le \bigoplus V_q(\mu)^{\oplus n(\mu)} \\ \quad \iff \text{if } m(\mu) > n(\mu), \text{ then } \mu < {}^{\exists}\nu \text{ s.t. } m(\nu) < n(\nu). \end{array} \right)$$

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${}^{\exists}\mathbf{ev}_a : U_q(L\mathfrak{g}) \to U_q(\mathfrak{g})$ ($a \in \mathbb{C}(q)^*$): evaluation map,

$\Longrightarrow$ minimal affinization of $V_q(\lambda) = \mathbf{ev}_a^* V_q(\lambda)$.

$(\because \mathbf{ev}_a^* V_q(\lambda) \cong V_q(\lambda) \oplus 0)$

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Graded limit

Let $M_q(\lambda)$ be a minimal affinization of $V_q(\lambda)$.

$M_q(\lambda)$: $U_q(L\mathfrak{g})$-module $\overset{q\to 1}{\Rightarrow}$ $M(\lambda)$: $L\mathfrak{g}$-module

From $M(\lambda)$, define a $\mathfrak{g}[t]$-module $M^{\mathrm{gr}}(\lambda)$ as follows:

$M(\lambda) \overset{\text{restriction}}{\to} M(\lambda)$: $\mathfrak{g}[t]$-module $(\mathfrak{g}[t] := \mathfrak{g} \otimes \mathbb{C}[t] \subseteq L\mathfrak{g})$

$\overset{\text{pull-back}}{\to} M^{\mathrm{gr}}(\lambda) := \tau_a^*\big(M(\lambda)\big)$: $\mathbb{Z}$-graded $\mathfrak{g}[t]$-module

$\left({}^{\exists}a \in \mathbb{C}; \tau_a(x \otimes t^k) = x \otimes (t+a)^k\right)$ graded limit

- $\mathfrak{g}$: type $ABC \Rightarrow M^{\mathrm{gr}}(\lambda)$: not depend on the choice of $M_q(\lambda)$.
- $\mathrm{ch}\, M_q(\lambda) = \mathrm{ch}\, M^{\mathrm{gr}}(\lambda)$,
- ($U_q(\mathfrak{g})$-multi.) $[M_q(\lambda) : V_q(\mu)] = [M^{\mathrm{gr}}(\lambda) : V(\mu)]$ ($\mathfrak{g}$-multi.).

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Main Theorem

$\hat{\mathfrak{g}} := \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d$: affine Lie algebra $\supseteq \mathfrak{g}[t]$,

$\hat{V}(\Lambda)$: irred. integrable h.w. $\hat{\mathfrak{g}}$-module with h.w. Λ,

For $w \in \hat{W}$, fix $0 \neq v_{w(\Lambda)} \in \hat{V}(\Lambda)_{w(\Lambda)}$: extremal weight vector

Assume $\mathfrak{g}$ is of type A_n, B_n or C_n.

$M_q(\lambda)$: minimal affiniz. of $V_q(\lambda) \rightsquigarrow M^{\mathrm{gr}}(\lambda)$: graded limit

Theorem

$\exists$ sequences $\Lambda^1, \dots, \Lambda^n$ and $w_1, \dots, w_n$ s.t.

$$M^{\mathrm{gr}}(\lambda) \cong U(\mathfrak{g}[t])(v_{w_1(\Lambda^1)} \otimes v_{w_2(\Lambda^2)} \otimes \cdots \otimes v_{w_n(\Lambda^n)})$$
$$\subseteq \hat{V}(\Lambda^1) \otimes \hat{V}(\Lambda^2) \otimes \cdots \otimes \hat{V}(\Lambda^n).$$

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Ex. Type B_n

$$\lambda = \sum_i \lambda_i \varpi_i \Rightarrow \Lambda^i = \begin{cases} \lambda_i \Lambda_1 & (i < n : \text{ odd}), \\ \lambda_i \Lambda_0 & (i < n : \text{ even}), \\ \lfloor \lambda_n / 2 \rfloor \Lambda_{\bar{n}} + \bar{\lambda}_n \Lambda_n & (i = n), \end{cases}$$

$$\left(\bar{n}, \bar{\lambda}_n \in \{0, 1\},\ n \equiv \bar{n},\ \lambda \equiv \bar{\lambda}_n \text{ (mod 2)} \right)$$

$$\tau_i := \begin{cases} s_1 s_2 \cdots s_i, & (i : \text{odd}), \\ s_0 s_2 \cdots s_i, & (i : \text{even}), \end{cases}$$

$$w_i := \tau_i \tau_{i+1} \cdots \tau_n$$

Corollaries

- $M^{\mathrm{gr}}(\lambda) \cong U(\mathfrak{g}[t])\big(v_{w_1(\Lambda^1)} \otimes v_{w_2(\Lambda^2)} \otimes \cdots \otimes v_{w_n(\Lambda^n)}\big) =: D(\lambda)$

$$\subseteq \hat{V}(\Lambda^1) \otimes \hat{V}(\Lambda^2) \otimes \cdots \otimes \hat{V}(\Lambda^n).$$

- Fact. $D(\lambda)$ has a character formula:

$$\mathrm{ch}\, D(\lambda) = \mathcal{D}_{w_0\tau_1}\Big(e(\Lambda^1) \cdot \mathcal{D}_{\tau_2}\Big(e(\Lambda^2) \cdots \mathcal{D}_{\tau_n}(e(\Lambda^n)) \cdots\Big)\Big)$$

$\Big(\hat{W} \ni \tau_i$ s.t. $w_i = \tau_i \tau_{i+1} \cdots \tau_n$, $\mathcal{D}_\tau$: Demazure op. $\Big)$

- Fact. $D(\lambda)$ has a crystal analog:

$$D(\lambda) \subseteq \hat{V}(\Lambda^1) \otimes \cdots \otimes \hat{V}(\Lambda^n) \overset{U_q(\hat{\mathfrak{g}})}{\rightsquigarrow} D_q(\lambda) \subseteq \hat{V}_q(\Lambda^1) \otimes \cdots \otimes \hat{V}_q(\Lambda^n)$$

$\Downarrow$ crystal basis $\Downarrow$

$${}^{\exists} D \subseteq B(\Lambda^1) \otimes \cdots \otimes B(\Lambda^n)$$

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$$D(\lambda) \subseteq \hat{V}(\Lambda^1) \otimes \cdots \otimes \hat{V}(\Lambda^n) \overset{U_q(\hat{\mathfrak{g}})}{\rightsquigarrow} D_q(\lambda) \subseteq \hat{V}_q(\Lambda^1) \otimes \cdots \otimes \hat{V}_q(\Lambda^n)$$

$\Downarrow$ crystal basis $\Downarrow$

$$^{\exists} D \subseteq B(\Lambda^1) \otimes \cdots \otimes B(\Lambda^n)$$

$$[D(\lambda) : V(\mu)] = \#\{b \in D \mid \mathrm{wt}(b) = \mu, \tilde{e}_i(b) = 0\ (1 \le i \le n)\}.$$

Corollaries

∘ $M^{\mathrm{gr}}(\lambda) \cong U(\mathfrak{g}[t])(v_{w_1(\Lambda^1)} \otimes v_{w_2(\Lambda^2)} \otimes \cdots \otimes v_{w_n(\Lambda^n)}) =: D(\lambda)$

$$\subseteq \hat{V}(\Lambda^1) \otimes \hat{V}(\Lambda^2) \otimes \cdots \otimes \hat{V}(\Lambda^n).$$

- Fact. $D(\lambda)$ has a character formula:

$$\mathrm{ch}\, M_q(\lambda) = \mathrm{ch}\, D(\lambda) = \mathcal{D}_{w_0\tau_1}\Big(e(\Lambda^1) \cdot \mathcal{D}_{\tau_2}\Big(e(\Lambda^2) \cdots \mathcal{D}_{\tau_n}(e(\Lambda^n)) \cdots\Big)\Big)$$

$$\Big(\hat{W} \ni \tau_i \text{ s.t. } w_i = \tau_i \tau_{i+1} \cdots \tau_n,\ \ \mathcal{D}_\tau\text{: Demazure op. }\Big)$$

- Fact. $D(\lambda)$ has a crystal analog:

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$${}^{\exists} D \subseteq B(\Lambda^1) \otimes \cdots \otimes B(\Lambda^n)$$

$$[M_q(\lambda) : V_q(\mu)] = \#\{b \in D \mid \mathrm{wt}(b) = \mu, \tilde{e}_i(b) = 0\ (1 \le i \le n)\}.$$